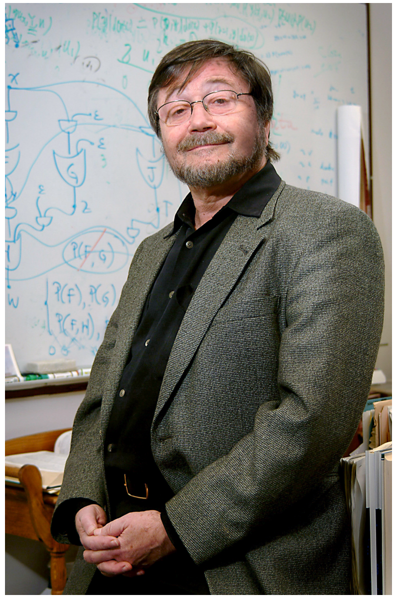

Artificial Intelligence

Constraint Satisfaction Problems (Part 2)

CS 444 – Spring 2021

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Much of this lecture is taken from
Dan Klein and Pieter Abbeel AI class at UC Berkeley

Today

- Review of A* Heuristics for PA 1
- Continue with Constraint Satisfaction problems
- Arc consistency AC-3 examples
- Problem Structure
- Min conflicts

Learning Objectives

- Apply the AC-3 to maintain arc consistency (MAC)
- Investigate the problem structure of CSPs for identify more efficient solutions using cutset conditioning and tree decomposition
- Apply min-conflicts algorithm and by able to code it to solve CSPs. Characterize the min-conflicts algorithm (runtime, completeness, etc).

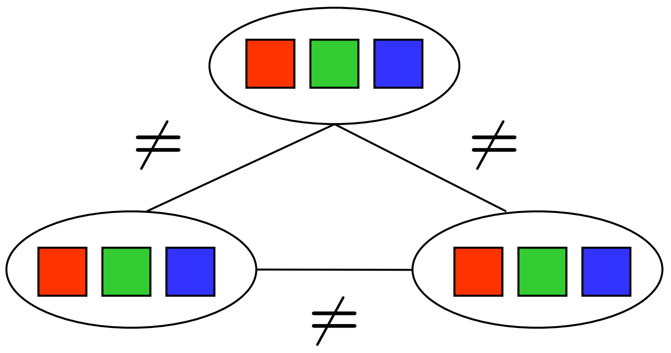
Figure from Berkley AI

Student Heuristic Presentation

- Alex Marasco – Finding/visit all the corners heuristics
- Garrett Christian -- Eat all the dots heuristic

CSP problems

- CSPs:
  - Variables
  - Domains
  - Constraints
    - Implicit (provide code to compute)
    - Explicit (provide a list of the legal tuples)
    - Types:
      - Unary (one variable)
      - Binary (two variables)
      - N-ary (n variables)



- Goals:
  - **In this class**: find any solution
  - Also: find all, find best, etc.

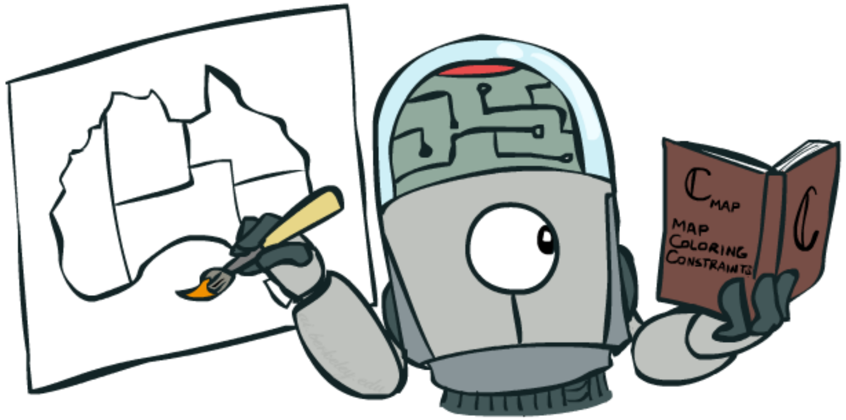


Figure from Berkley AI

Backtracking Search

```
function BACKTRACKING-SEARCH(csp) returns solution/failure
    return RECURSIVE-BACKTRACKING({ }, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
        if value is consistent with assignment given CONSTRAINTS[csp] then
            add {var = value} to assignment
            result ← RECURSIVE-BACKTRACKING(assignment, csp)
            if result ≠ failure then return result
            remove {var = value} from assignment
    return failure
```

Backtracking Example

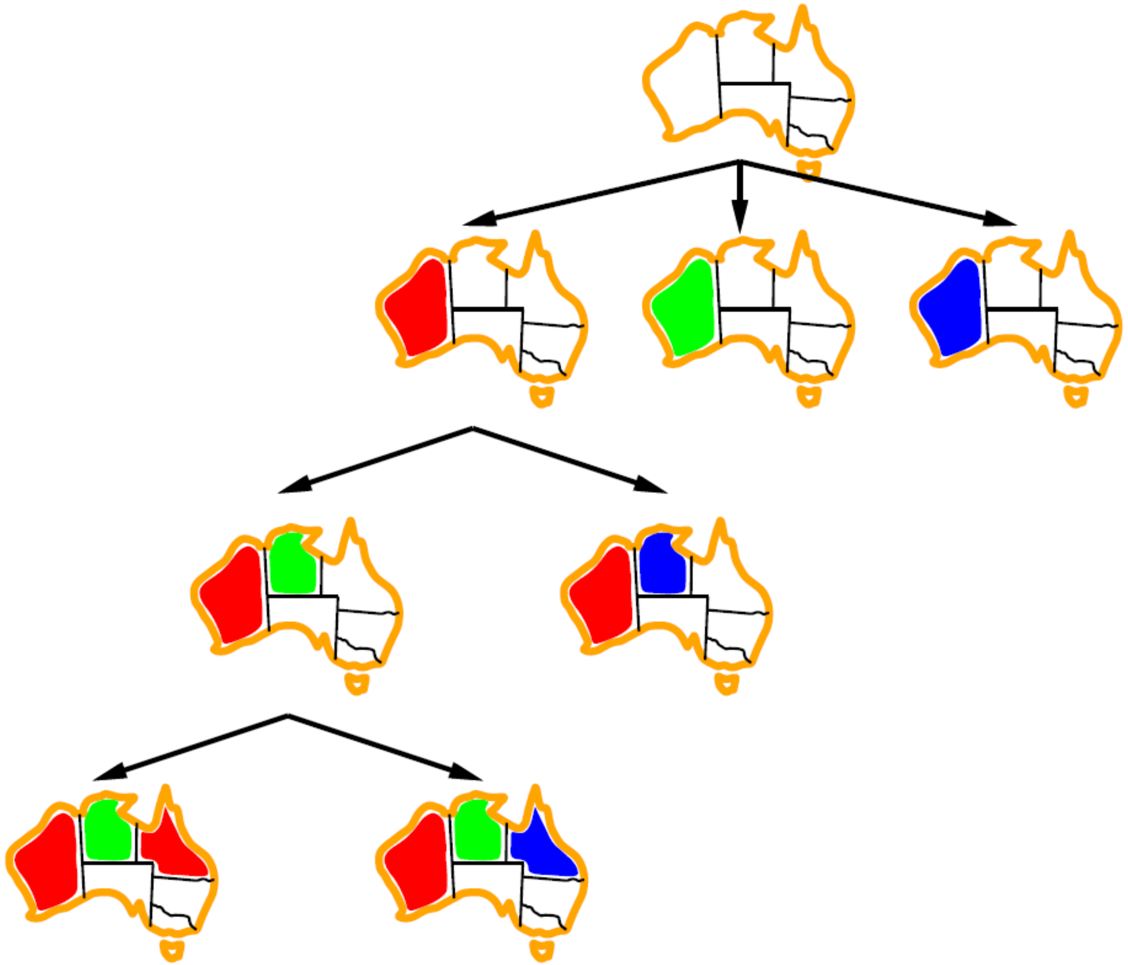


Figure from Berkley AI

Improving Backtracking

- General-purpose ideas give huge gains in speed
  - … but it's all still **NP-hard**

- **Filtering**: Can we detect inevitable failure early?

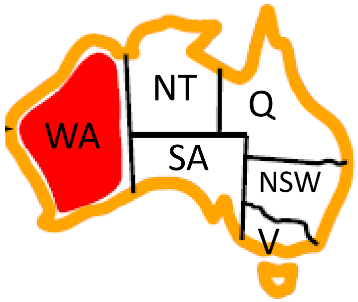
- **Ordering**:
  - Which **variable** should be assigned next? (MRV)
  - In what order should its **values** be tried? (LCV)

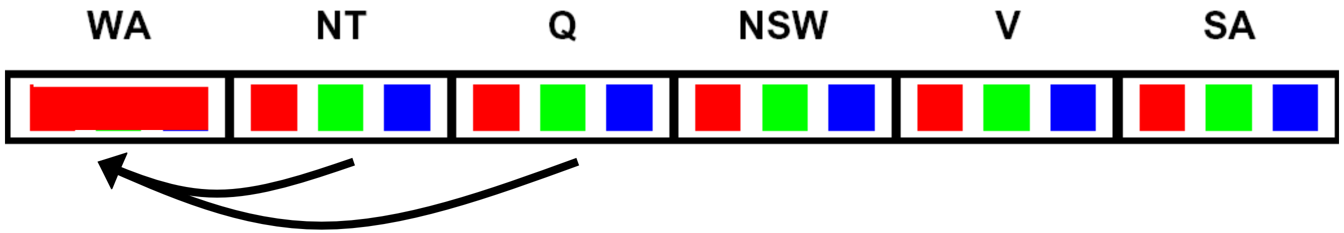
- **Structure**: Can we exploit the problem structure?

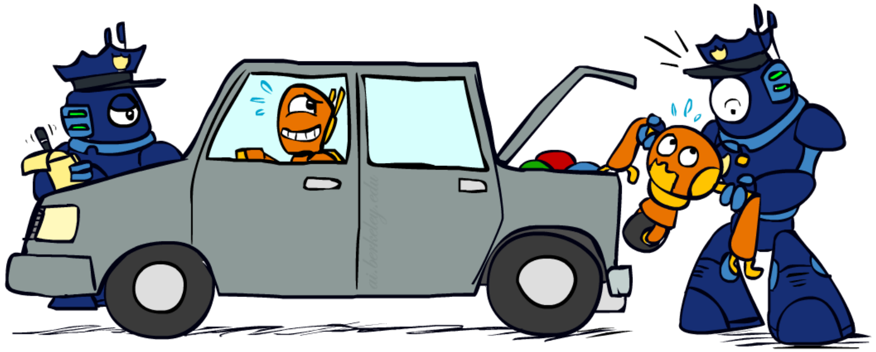
Figure from Berkley AI

Consistency of a Single Arc

- An arc X → Y is consistent iff for *every* x in the tail there is *some* y in the head which could be assigned without violating a constraint





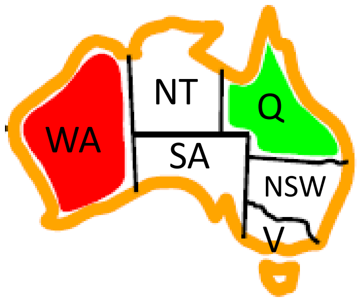


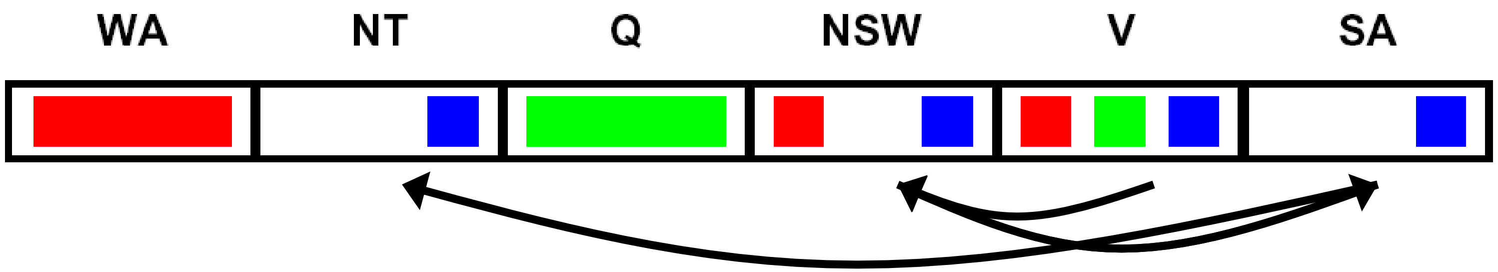
Delete from the tail!

- Forward checking: Enforcing consistency of arcs pointing to each new assignment

Arc Consistency of an Entire CSP

- A simple form of propagation makes sure **all** arcs are consistent:





- Important: If X loses a value, neighbors of X need to be rechecked!
- Arc consistency detects failure **earlier** than forward checking
- Can be run as a preprocessor or after each assignment
- What's the downside of enforcing arc consistency?

Remember: Delete from the tail!

Figure from Berkley AI

Enforcing Arc Consistency in a CSP

```
function AC-3( csp) returns the CSP, possibly with reduced domains
   inputs: csp, a binary CSP with variables {X1, X2, ..., Xn}
   local variables: queue, a queue of arcs, initially all the arcs in csp

   while queue is not empty do
      (Xi, Xj) ← REMOVE-FIRST(queue)
      if REMOVE-INCONSISTENT-VALUES(Xi, Xj) then
         for each Xk in NEIGHBORS[Xi] do
            add (Xk, Xi) to queue
```

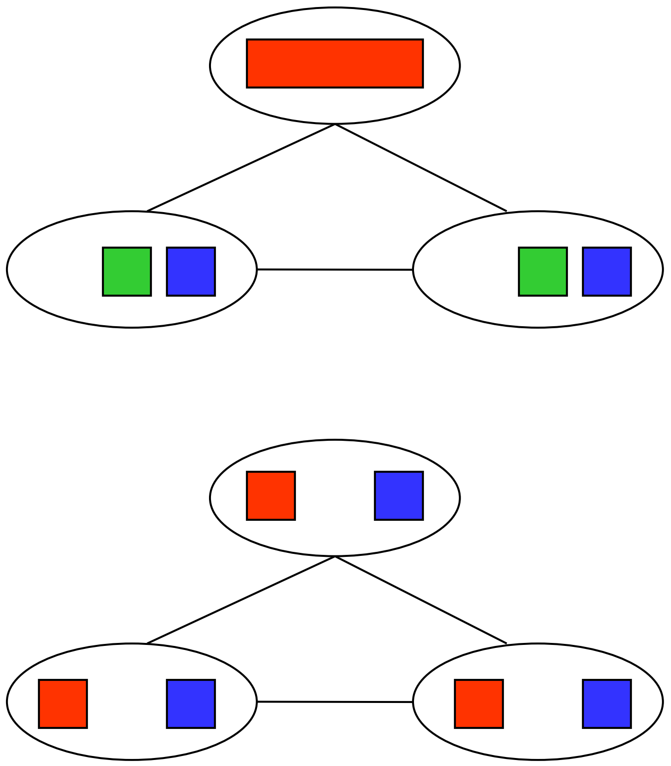
```
function REMOVE-INCONSISTENT-VALUES( Xi, Xj) returns true iff succeeds
   removed ← false
   for each x in DOMAIN[Xi] do
      if no value y in DOMAIN[Xj] allows (x,y) to satisfy the constraint Xi ↔ Xj
         then delete x from DOMAIN[Xi]; removed ← true
   return removed
```

- Runtime: $O(n^2d^3)$, can be reduced to $O(n^2d^2)$
- … but detecting all possible future problems is NP-hard – why?

Limitations of Arc Consistency

- After enforcing arc consistency:
  - Can have one solution left
  - Can have multiple solutions left
  - Can have no solutions left (and not know it)

- Arc consistency still runs inside a backtracking search!



What went wrong here?

Figure from Berkley AI

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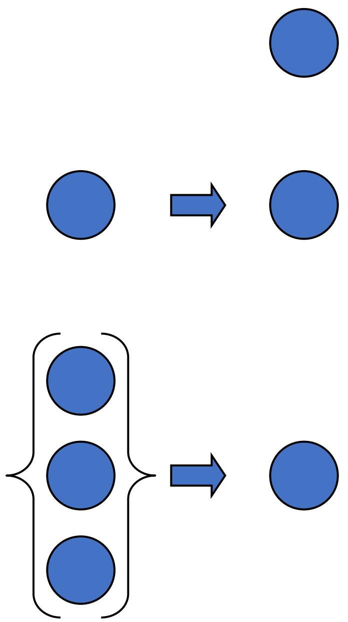
K-Consistency

- Increasing degrees of consistency
  - 1-Consistency (Node Consistency): Each single node's domain has a value which meets that node's unary constraints
  - 2-Consistency (Arc Consistency): For each pair of nodes, any consistent assignment to one can be extended to the other
  - K-Consistency: For each k nodes, any consistent assignment to k-1 can be extended to the k^{th} node.



- Higher k more expensive to compute
- (You need to know the k=2 case: arc consistency)

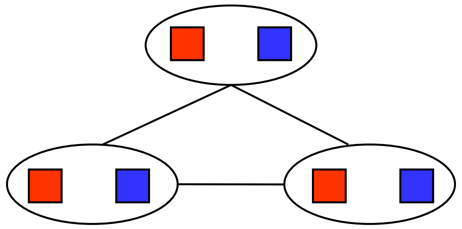


Figure from Berkley AI

Strong K-Consistency

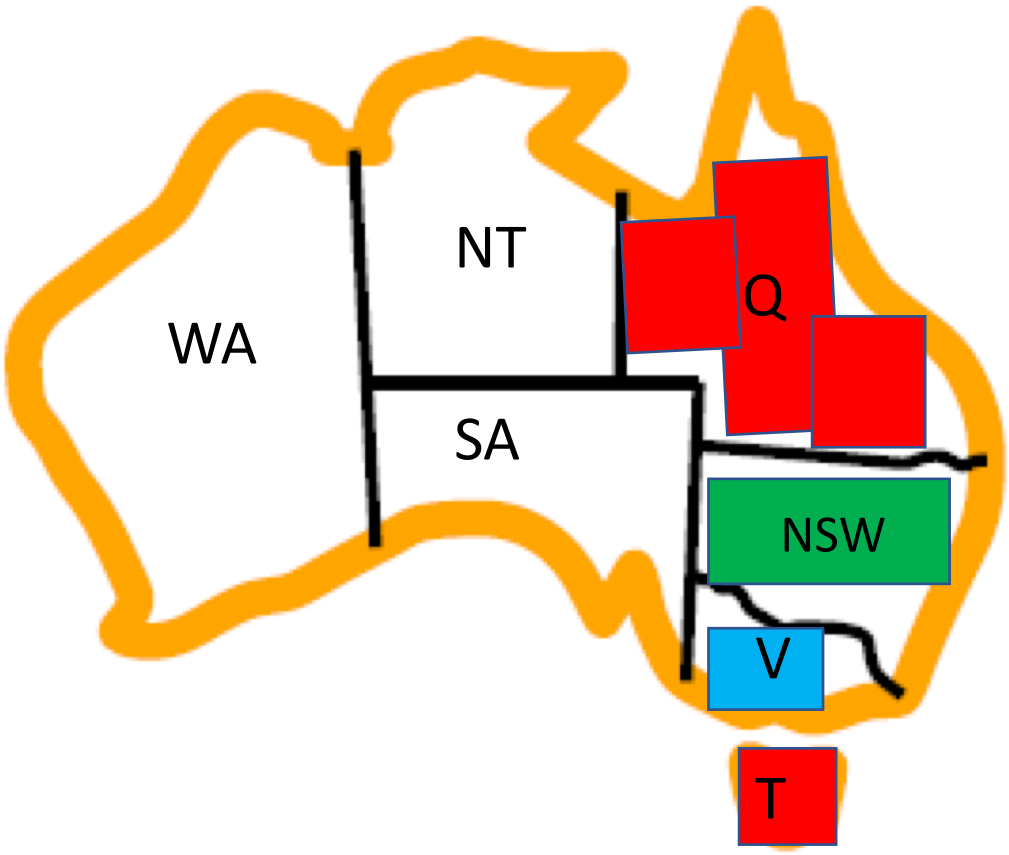
- Strong k-consistency: also k-1, k-2, … 1 consistent

- Claim: strong n-consistency means we can solve without backtracking!

- Why?
  - Choose any assignment to any variable
  - Choose a new variable
  - By 2-consistency, there is a choice consistent with the first
  - Choose a new variable
  - By 3-consistency, there is a choice consistent with the first 2
  - …

- Lots of middle ground between arc consistency and n-consistency! (e.g. k=3, called path consistency)

Intelligent Backtracking

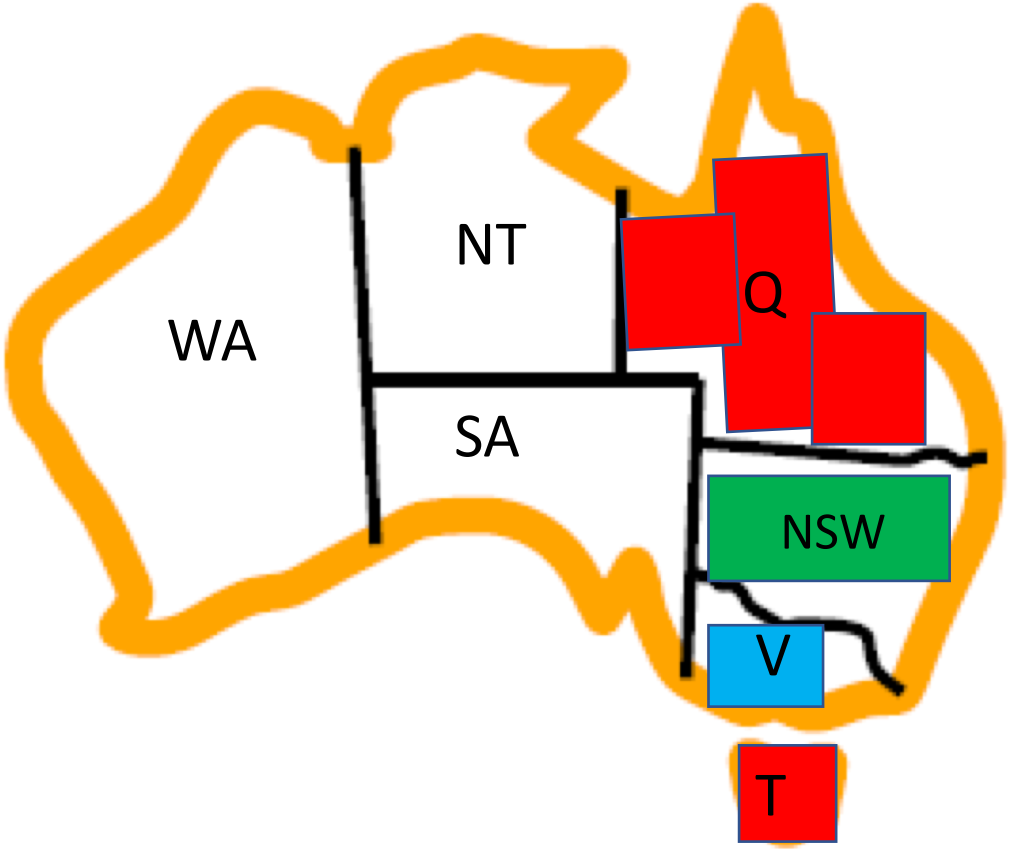


Variable assignment order: {Q, NSW, V, T, SA, WA, NT}

Partial Assignment: {Q = red, NSW=green, V=blue, T=red}

What does normal backtracking do when it tries to assignment **SA** a color?

Intelligent Backtracking



Variable assignment order: {Q, NSW, V, T, SA, WA, NT}

Partial Assignment: {Q = red, NSW=green, V=blue, T=red}

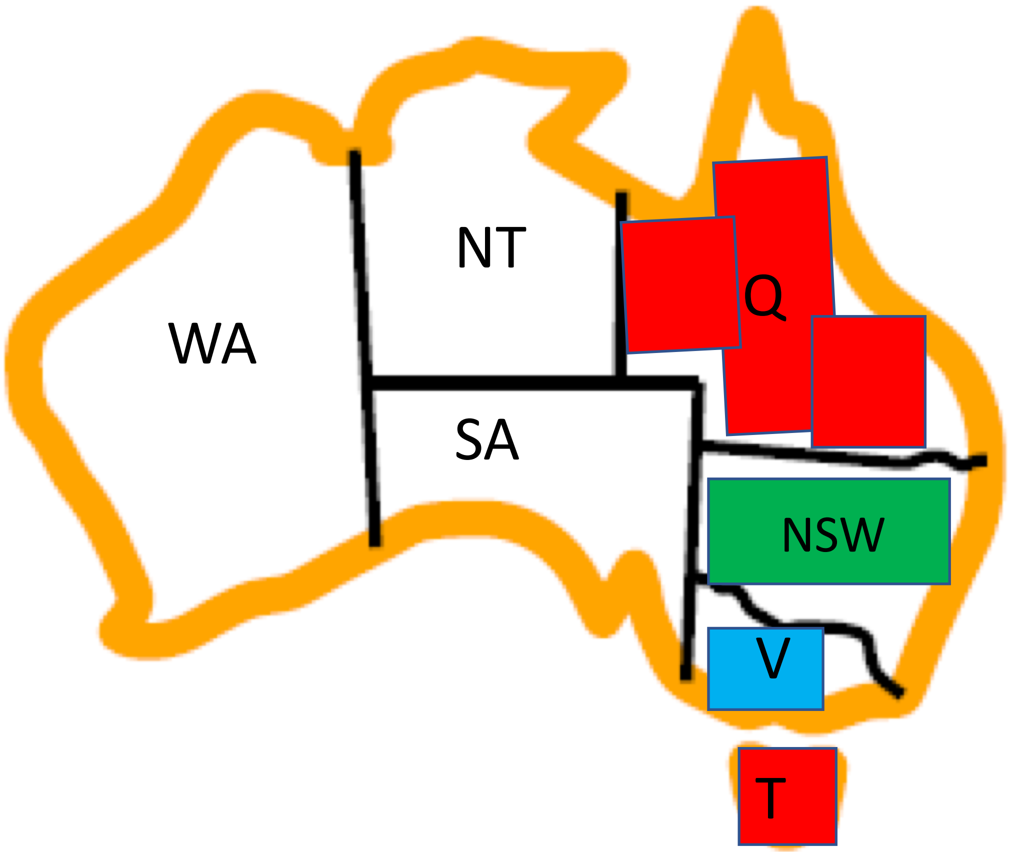
What does normal backtracking do when it tries to assignment **SA** a color?

- It tries all 3 colors. None of these work. So backtrack
- Change the color of T and try SA again
- Still no assignment works for SA. So backtrack.
- Etc.

How can we make this better?

Figure from Berkley AI

Intelligent Backtracking – Back jumping



Variable assignment order: {Q, NSW, V, T, SA, WA, NT}

Partial Assignment: {Q = red, NSW=green, V=blue, T=red}

Idea: Jump to a variable that is causing a problem.

Define a **conflict set**, which is built as we evaluate a variable. So, for SA, we check:

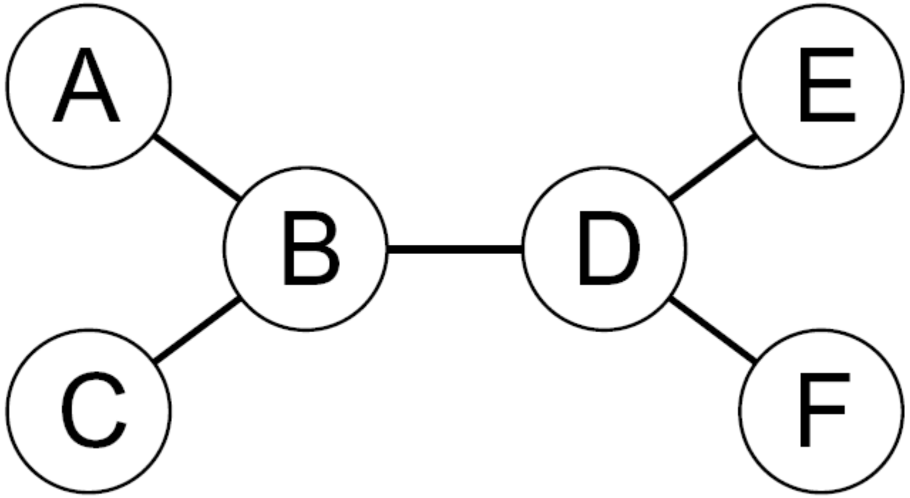
- Can't use red, Q is added to the conflict set for SA.
- Can't use green, NSW is added to the conflict set for SA.
- Can't use blue, V is added to the conflict set for SA.

Backtrack to at least one of these variables so we have a chance of correcting the issue.

https://inst.eecs.berkeley.edu/~cs188/fa19/assets/demos/csp/csp_demos.html

Figure from Berkley AI

Tree-Structured CSPs

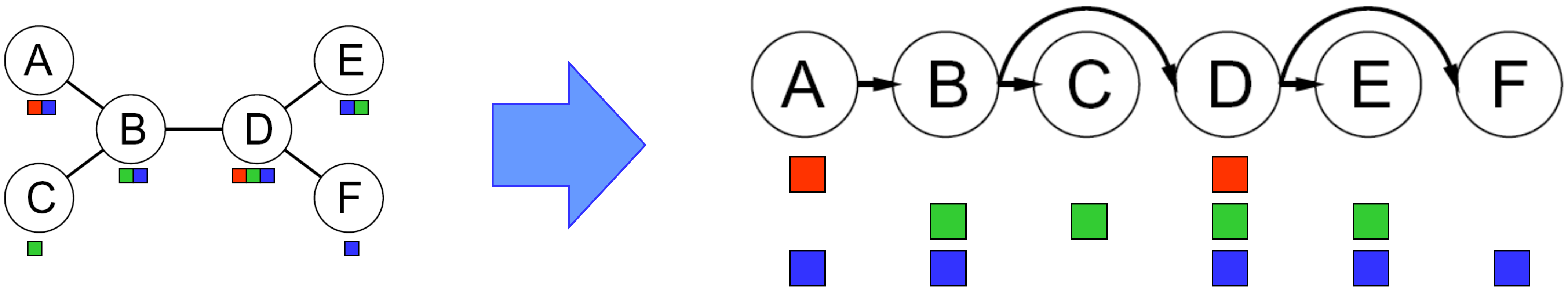


- Theorem: if the constraint graph has no loops, the CSP can be solved in $O(n\ d^2)$ time
  - Compare to general CSPs, where worst-case time is $O(d^n)$

- This property also applies to probabilistic reasoning (later): an example of the relation between syntactic restrictions and the complexity of reasoning

Tree-Structured CSPs

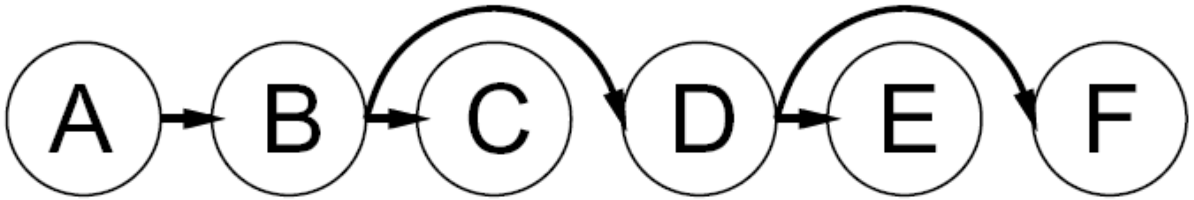
- Algorithm for tree-structured CSPs:
  - Order: Choose a root variable, order variables so that parents precede children



  - Remove backward: For i = n : 2, apply RemoveInconsistent(Parent(X_i),X_i)
  - Assign forward: For i = 1 : n, assign X_i consistently with Parent(X_i)
- Runtime: O(n d^2) (why?)

Tree-Structured CSPs

- Claim 1: After backward pass, all root-to-leaf arcs are consistent
- Proof: Each X→Y was made consistent at one point and Y's domain could not have been reduced thereafter (because Y's children were processed before Y)



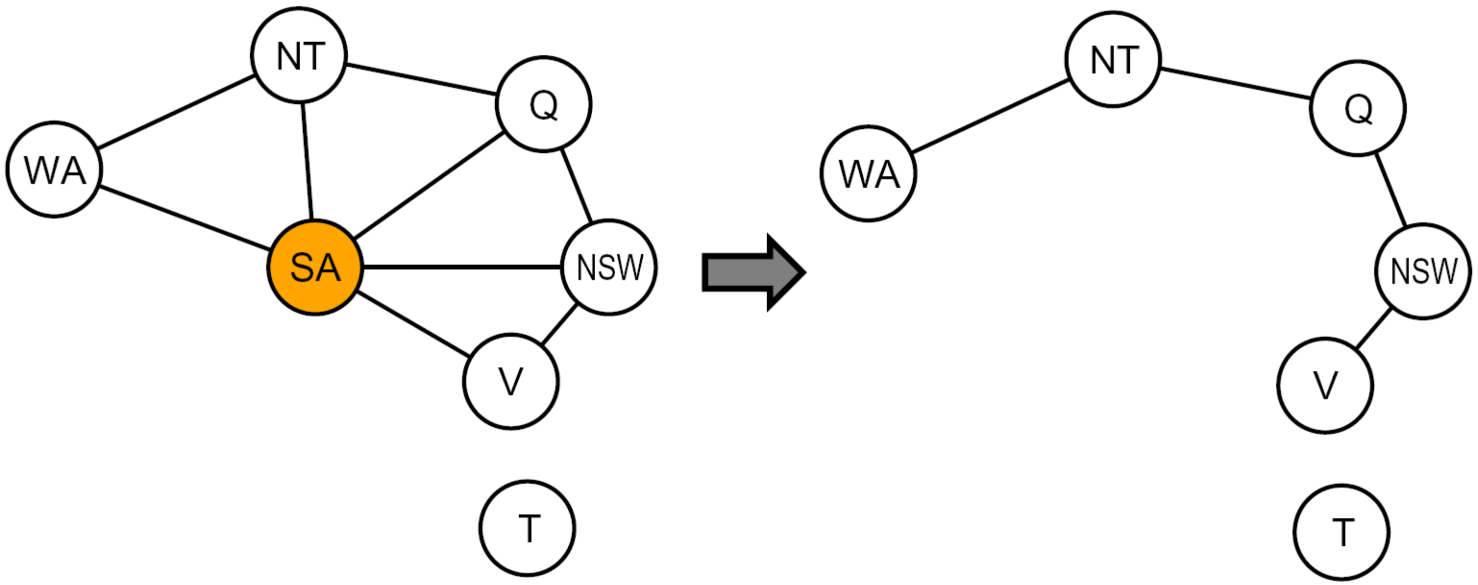
- Claim 2: If root-to-leaf arcs are consistent, forward assignment will not backtrack
- Proof: Induction on position

- Why doesn't this algorithm work with cycles in the constraint graph?

- Note: we'll see this basic idea again with Bayes' nets

Figure from Berkley AI

Nearly Tree-Structured CSPs



- **Conditioning**: instantiate a variable, prune its neighbors' domains
- **Cutset conditioning**: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree
- Cutset size c gives runtime $O((d^c) (n-c) d^2)$, very fast for small c

Figure from Berkley AI

Cutset Conditioning

Choose a cutset

Instantiate the cutset (all possible ways)

Compute residual CSP for each assignment

Solve the residual CSPs (tree structured)

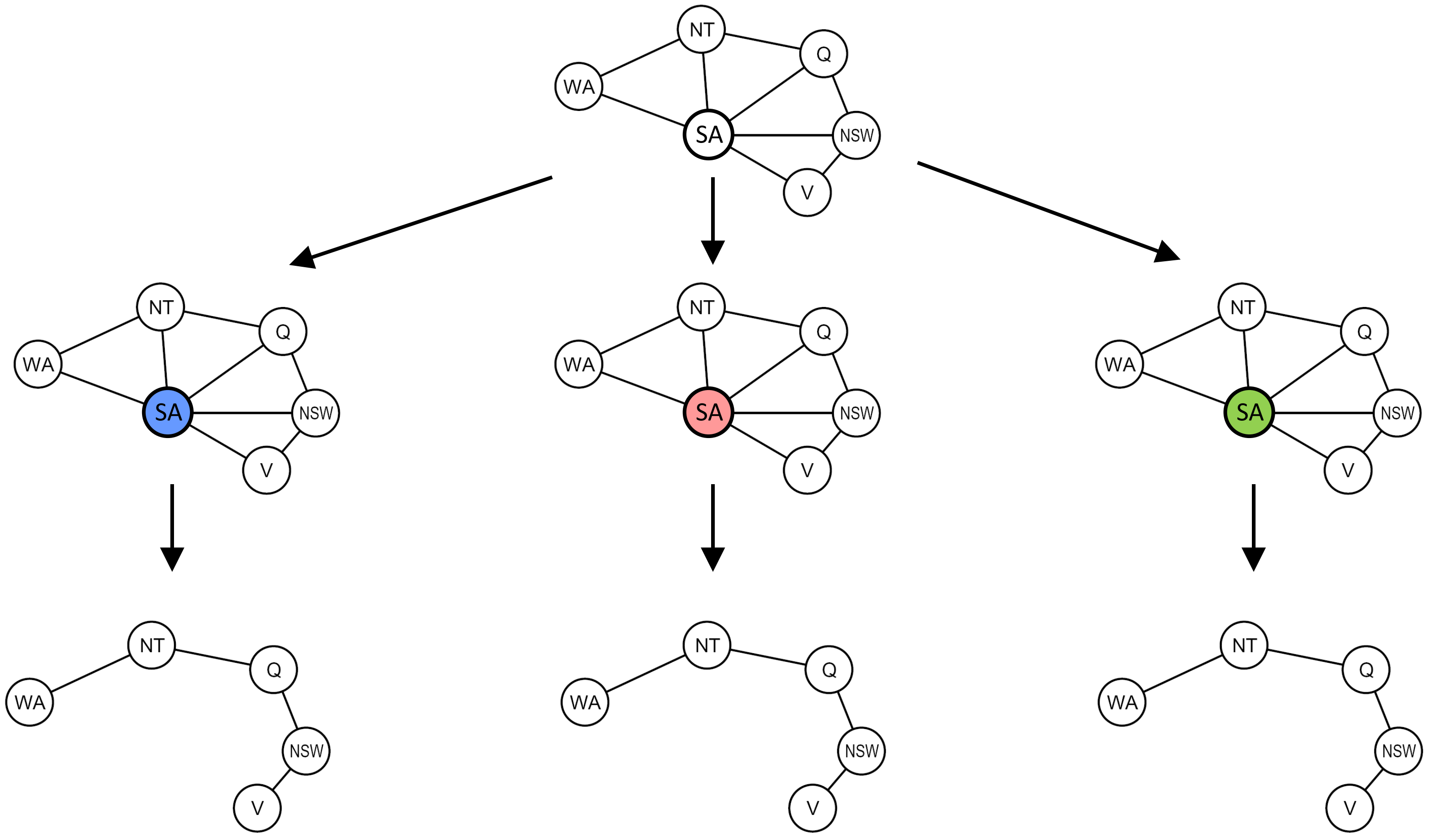


Figure from Berkley AI

Cutset Quiz

- Find the smallest cutset for the graph below.

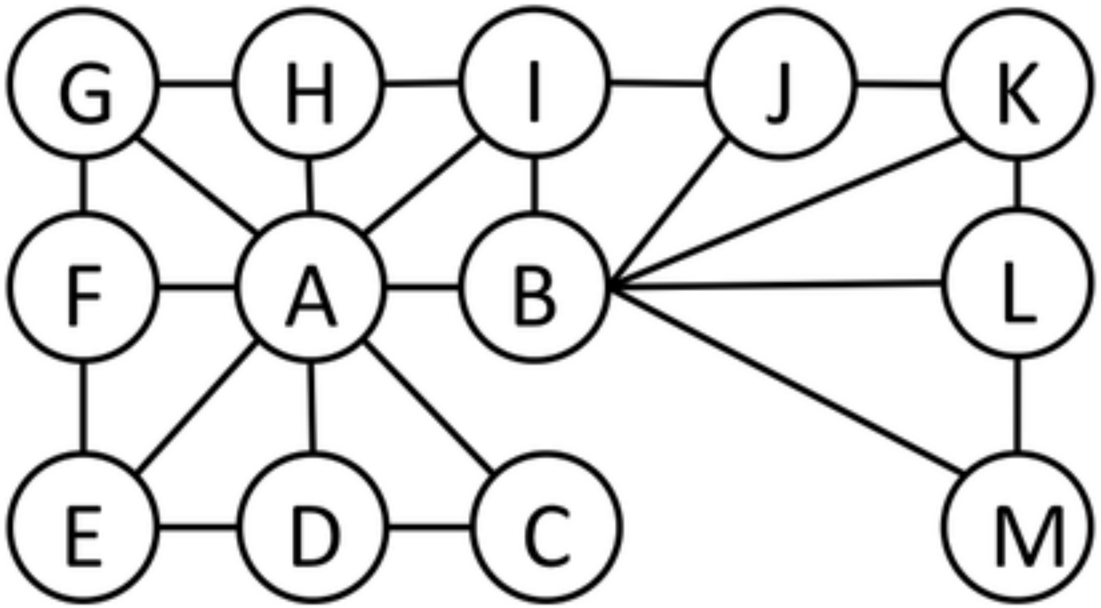
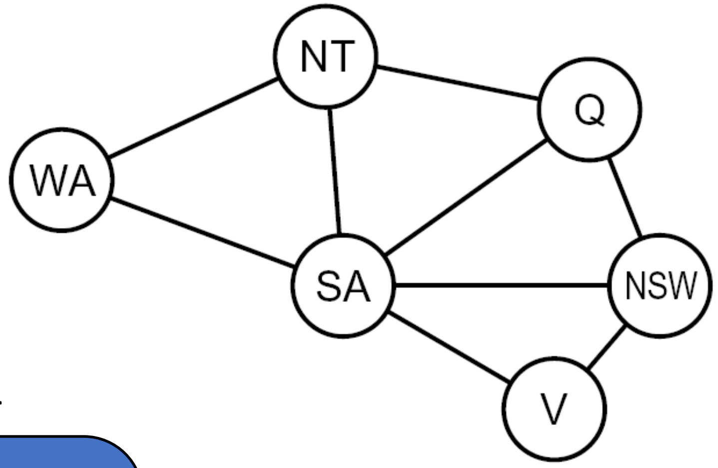
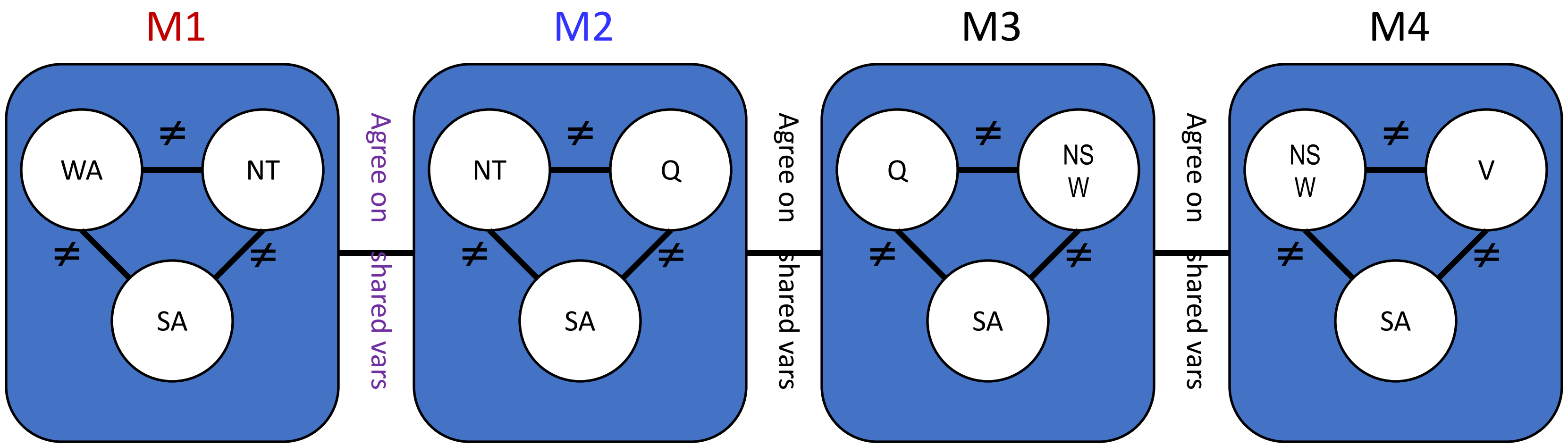


Figure from Berkley AI

Tree Decomposition

- Idea: create a tree-structured graph of mega-variables
- Each mega-variable encodes part of the original CSP
- Subproblems overlap to ensure consistent solutions





{(WA=r,SA=g,NT=b),
(WA=b,SA=r,NT=g),
…}

{(NT=r,SA=g,Q=b),
(NT=b,SA=g,Q=r),
…}

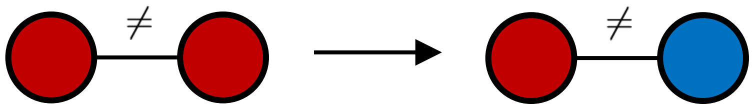
Agree: (M1,M2) ∈
{((WA=g,SA=g,NT=g), (NT=g,SA=g,Q=g)), …}

Figure from Berkley AI

Iterative Algorithms for CSPs

- Local search methods typically work with "complete" states, i.e., all variables assigned

- To apply to CSPs:
  - Take an assignment with unsatisfied constraints
  - Operators *reassign* variable values
  - No fringe! Live on the edge.



- Algorithm:

While not solved:
  Variable selection: randomly select any conflicted variable

  Value selection: min-conflicts heuristic:
    Choose a value that violates the fewest constraints
    I.e., hill climb with h(n) = total number of violated constraints

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Figure from Berkley AI

Performance of Different CSP Algorithms

Problem	Backtracking	BT+MRV	Forward Checking	FC+MRV	Min-Conflicts
USA (4 color)	(> 1,000,000)				
n-Queens	(> 40,000,000)				

Figure from Berkley AI

Performance of Different CSP Algorithms

Problem	Backtracking	BT+MRV	Forward Checking	FC+MRV	Min-Conflicts
USA (4 color)	(> 1,000,000)	(> 1,000,000)			
n-Queens	(> 40,000,000)	13,500,000			

Figure from Berkley AI

Performance of Different CSP Algorithms

Problem	Backtracking	BT+MRV	Forward Checking	FC+MRV	Min-Conflicts
USA (4 color)	(> 1,000,000)	(> 1,000,000)	2,000		
n-Queens	(> 40,000,000)	13,500,000	(> 40,000,000)		

Figure from Berkley AI

Performance of Different CSP Algorithms

Problem	Backtracking	BT+MRV	Forward Checking	FC+MRV	Min-Conflicts
USA (4 color)	(> 1,000,000)	(> 1,000,000)	2,000	60	
n-Queens	(> 40,000,000)	13,500,000	(> 40,000,000)	817,000	

Figure from Berkley AI

Performance of Different CSP Algorithms

Problem	Backtracking	BT+MRV	Forward Checking	FC+MRV	Min-Conflicts
USA (4 color)	(> 1,000,000)	(> 1,000,000)	2,000	60	64
n-Queens	(> 40,000,000)	13,500,000	(> 40,000,000)	817,000	4,000

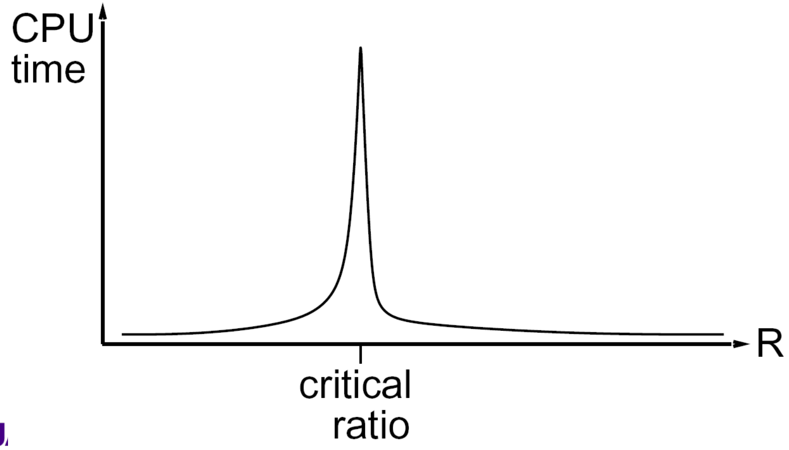
Figure from Berkley AI

Don't Make Things too Complicated

- Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000)!

- The same appears to be true for any randomly-generated CSP *except* in a narrow range of the ratio

$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$



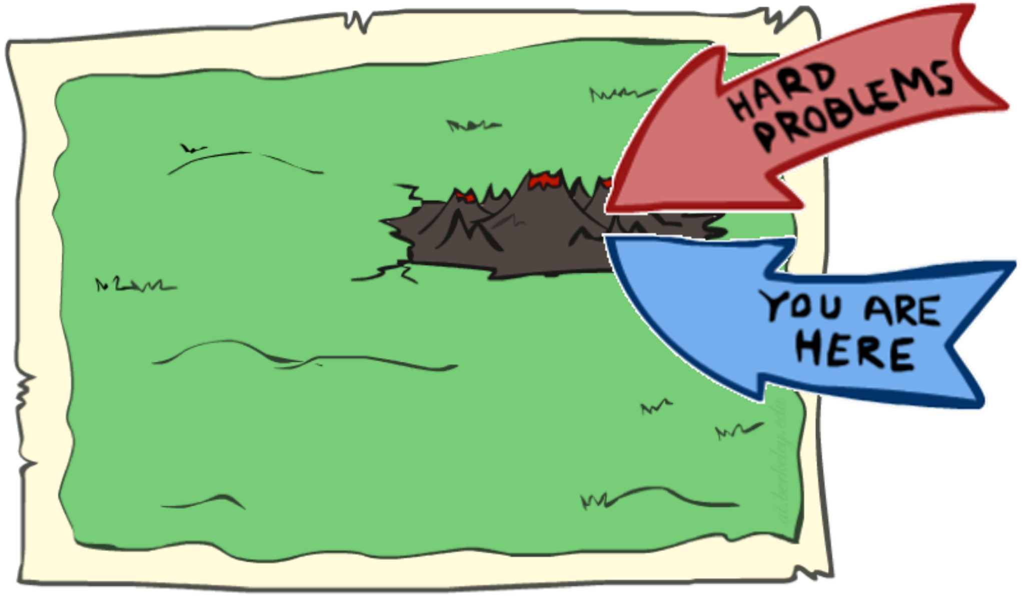


Figure from Berkley AI

CSP Summary

- CSPs are a special kind of search problem:
  - States are partial assignments
  - Goal test defined by constraints
- Basic solution: backtracking sear
- Speed-ups:
  - Ordering
  - Filtering
  - Structure
- Iterative min-conflicts is often effective in practice

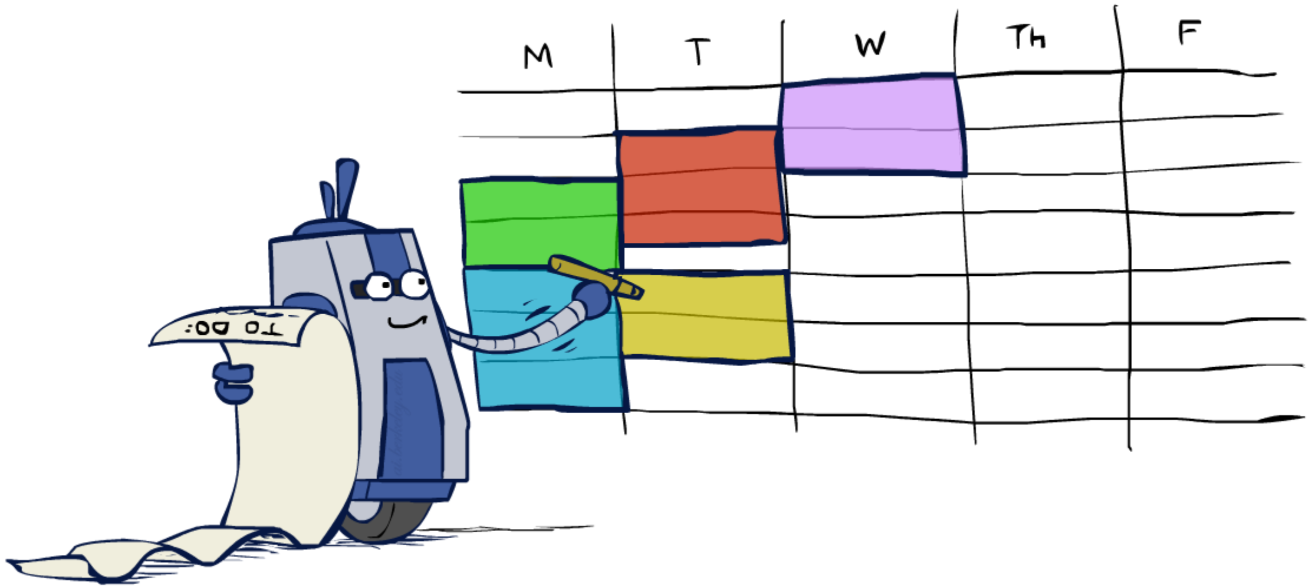


Figure from Berkley AI